# A tutorial on Analog Computation

## Computing functions over the reals

## I

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Tutorial outline:

I Computing real functions

II Continuous dynamical systems and computation

III Real recursive functions

# Outline

Is $f :\subset \mathbb{R} \to \mathbb{R}$ computable?

Several notions of computability for real functions:

- Turing machine approach: Computable Analysis
- Continuous time analog models
- BSS machines
- ...

# The differential analyser

Computing in continuous time with an analog computer (a first example):

> *The differential analyser, which concept dates to Lord Kelvin and his brother James Thomson in 1876, and was constructed at MIT under the supervision of Vannevar Bush in 1932.*

# The differential analyser

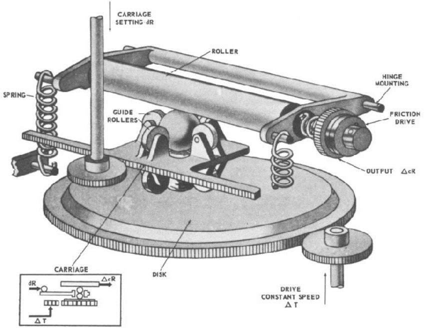


**Figure:** A mechanical integrator (Bureau of Naval Personnel, *Basic Machines and How They Work*, 1964)

# Analog circuits

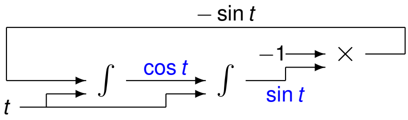


**Figure:** A circuit that calculates sin and cos. Its initial conditions are $\sin(0) = 0$ and $\cos(0) = 1$. The output $w$ of the integrator unit $\int$ obeys $dw = u\,dv$ where $u$ and $v$ are its upper and lower inputs respectively.

The circuit above is represented by the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

which solution is $x_1 = \sin t$ and $x_2 = \cos t$ given the initial conditions $x_1(0) = 0$ and $x_2(0) = 1$.

# Shannon's GPAC

In 1941 Claude Shannon proposed a mathematical model for the Differential Analyser.

Shannon proved that, given a sufficient number of integrators, any differentially algebraic function, i.e., a solution of

$$p(t, x, x', \ldots, x^{(n)}) = 0$$

could be generated.

Shannon's model was refined by Graça and Costa (2003) who showed that all non-degenerate GPAC functions are precisely PIVP functions, i.e. solutions of polynomial initial value problems

$$\bar{x}' = p(\bar{x}, t) \quad , \quad \bar{x}(0) = \bar{x}_0.$$

## Computable real functions

*What is a computable real function?*

Since real numbers and many other objects studied in analysis are “infinite” objects containing an “infinite amount of information”, one has to approximate them by “finite” objects containing only a “finite amount of information” and to perform the actual computations on these finite objects. (Braktta *et al.*, A Tutorial on Computable Analysis, 2008)

For instance, a real number is computable if there is a Turing machine with no input that outputs a binary expansion of that number (note: the output tape is one-way).

# Computable reals

**Definition (Cauchy representation)**

A sequence $\{r_n\}$ of rationals is a $\rho$-name of a real number $x$ if there exists three functions $a, b, c$ from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$

$$r_n = (-1)^{a(n)} \frac{b(n)}{c(n)+1} \text{ and } |r_n - x| \leq 2^{-n}.$$

**Definition (Computable real number)**

$x \in \mathbb{R}$ is computable if it has a computable $\rho$-name, i.e., if $a$, $b$ and $c$ are computable.

$M$ is an oracle Turing machine if, at any step of the computation of $M$ using oracle $\phi : \mathbb{N} \to \mathbb{N}^k$, $M$ is allowed to query the value of $\phi(n)$ for any $n$. (Below, $\phi$ is going to be a $\rho$-name for $x$.)

**Definition (Computable function)**

A function $f : D \subset \mathbb{R}^m \to \mathbb{R}^p$ is computable if there is an oracle Turing machine such that for any accuracy $n$ and any $\rho$-name for $x \in D$ given as oracle, computes a rational vector $r$ satisfying $||r - f(x)|| \leq 2^{-n}$.

In other words, the machine produces a rapidly converging rational sequence with limit $f(x)$.

**Definition**

$\mathbf{C}(\mathbb{R})$ denotes the set of computable functions.

# Are distinct notions of computability comparable?

$\mathbf{C}(\mathbb{R})$ = *“Analog”?*

Analog models:

- Polynomial Differential Equations/Shannon’s GPAC
- Real Recursive Functions
- ...

## GPAC generable functions

**Definition (PIVP function with parameters in $S$)**

$\bar{x}$ (or $x_i$) is a PIVP function with parameters in $S$ if $\bar{x}$ is the solution of a polynomial initial value problem $\bar{x}' = p(\bar{x}, t)$, $\bar{x}(0) = \bar{x}_0$ where the coefficients of $p$ and the components of $\bar{x}_0$ are in $S$.

**Theorem (follows from Graça, Zhong and Buescu, 2007)**

*Let $\bar{x} : (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^k$ be a PIVP function with computable parameters. Then, $\bar{x}$ is computable on $(\alpha, \beta)$.*

But there are functions which are computable and are not PIVP.

**Theorem (see *e.g.* Rubel, 1989)**

*The Gamma function $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is not differentially algebraic.*

# Approximation and limit

*Is the GPAC too weak, or is its notion of computability inadequate?*

There is a kind of limit which is implicit in the definitions of computable analysis.

**Definition (Approximation)**

Consider functions $f$ and $f^*$. We write $f(\bar{x}) \preceq f^*(\bar{x}, t)$, if $|f(\bar{x}) - f^*(\bar{x}, t)| < \frac{1}{t}$, for all $\bar{x}$ in the domain of $f$, and all $t > 0$.

**Definition (LIM)**

LIM is an operation which takes a function $f^*$ and returns $f(\bar{x}) = \lim_{t\to\infty} f^*(\bar{x}, t)$ as long as the limit exists and $f(\bar{x}) \preceq f^*(\bar{x}, t)$.

# The class of real computable functions is closed under LIM

**Theorem**

*If $f^* \in \mathbf{C}(\mathbb{R})$ and $f = LIM(f^*)$, then $f \in \mathbf{C}(\mathbb{R})$.*

**Proof.**

$f^* \in \mathbf{C}(\mathbb{R}) \implies$ there is a computable $\{r_n\}$ s.t. $|r_n - f^*(x,t)| < \frac{1}{n}$
$f = \mathrm{LIM}(f^*) \implies |f(x) - f^*(x,n)| < \frac{1}{n}$
Therefore,
$|r_{2n} - f(x)| = |r_{2n} - f^*(x,2n) + f^*(x,2n) - f(x)| \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$. $\square$

# Approximations between classes

**Definition**

For classes of functions $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \preceq \mathcal{B}$ if for any $f$ in $\mathcal{A}$ there is some $f^*$ in $\mathcal{B}$ such that $f \preceq f^*$.

**Theorem (Transitivity)**

*Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are classes of functions and suppose $\mathcal{C}$ contains $+$, $\mathrm{id}$ and is closed under composition. Then $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{C}$ implies $\mathcal{A} \preceq \mathcal{C}$.*

**Theorem**

*If $\mathcal{A} \preceq \mathcal{B}$ then $\mathcal{A} \subseteq \mathcal{B}(LIM)$*

where $\mathcal{B}(\mathrm{LIM})$ is the closure of $\mathcal{B}$ under LIM.

# A framework for analog computability

Let $A$ represent the set of functions that some analog model defines.

Goal. $\mathbf{C}(\mathbb{R}) = \mathrm{A}(\mathrm{LIM})$

The proof can be broken into 2 steps:

- (Approximation) $\mathbf{C}(\mathbb{R}) \approx \mathrm{A}$.
- (Completion) $\mathbf{C}(\mathbb{R}) = \mathrm{A}(\mathrm{LIM})$.

## GPAC-computability

Graça (2004) proposed a new notion of computability with the GPAC, under which the Gamma function becomes computable.

**Definition (GPAC-computability)**

A function $f : [0, 1] \to \mathbb{R}$ is GPAC−computable iff there exists some computable polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $q : \mathbb{R} \to \mathbb{R}$, and $n - 1$ computable real values $\alpha_1, ..., \alpha_{n-1}$ such that:

1. $(y_1, ..., y_n)$ is the solution of Cauchy problem $y' = p(y, t)$ with initial condition $(\alpha_1, ..., \alpha_{n-1}, q(x))$ set at time $t_0 = 0$
2. There are $i, j \in \{1, ..., n\}$ such that $\lim_{t\to\infty} y_j(t) = 0$ and $|f(x) - y_i(t)| \leq y_j(t)$ for all $x \in [0, 1]$ and all $t \in [0, +\infty)$.

## GPAC-computability

Consider the “PIVP functions” with a second argument, the initial value $x \in \mathbb{R}$:

### Definition

Let PI be the operation:

- Input: $n-1$ polynomials: $\bar{p}(y, t)$, a polynomial $q(x)$, and initial values $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$.
- Output: $y_1(t, x)$ where $(y_1, ..., y_n)$ is the solution of IVP: $\frac{\partial}{\partial t}\bar{y} = \bar{p}(\bar{y}, t)$ $\bar{y}(0) = (\alpha_1, ..., \alpha_{n-1}, q(x))$

### Definition

For $X \subseteq \mathbb{R}$, let $\mathrm{GPAC}_X$ be the set of functions generated by PI using polynomials with coefficients from $X$ and initial values from $X$.

## GPAC-computability

Consider the algebraic ring extension of $\mathbb{Q}$ by adjoining $\pi$ (it is the smallest ring that contains all rationals and $\pi$):

$$\mathbb{Q}[\pi] := \{a_n\pi^n + \cdots + a_1\pi + a_0 \in \mathbb{R} \mid a_0, \ldots, a_n \in \mathbb{Q}\}.$$

**Theorem (see Bournez, Campagnolo, Graça and Hainry, 2007)**

*On computable compact intervals:*

- *(Approximation)* $\mathbf{C}(\mathbb{R}) \approx GPAC_{\mathbb{Q}[\pi]}$
- *(Completion)* $\mathbf{C}(\mathbb{R}) = GPAC_{\mathbb{Q}[\pi]}(LIM^*)$

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