A tutorial on Analog Computation

Computing functions over the reals

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Tutorial outline:

I Computing real functions

II Continuous dynamical systems and computation

III Real recursive functions

Outline

Motivation

- How can we link computation and continuous dynamical systems?
- What does this tell us about continuous dynamical systems?
- Is there a canonical continuous dynamical system with respect to computability?

Polynomial initial value problems

- Common: they define most of the usual mathematical functions, in particular the "elementary functions" of Analysis;
- Widely used: e.g. Lorenz, Lokta-Volterra, or Van der Pol equations;
- Challenging: many open questions, e.g. the second part of Hilbert's 16th problem is to decide an upper bound for the number of limit cycles in polynomial vector fields of degree n.
- They satisfy an elimination result; Given $x' = f(x, t)$ and $x(0) = x_0$, with f a composition of polynomials and PIVP functions, x is given by the first components of some PIVP function.

Polynomial initial value problems

Some particular questions for PIVPs:

- Are PIVP functions computable?
- Is the domain of the solution computable?
- Is it even decidable if the domain is bounded?
- Is the reachability problem for PIVPs decidable?

Overview

$$\begin{array}{ccccc} \text{Unbounded domain} & & \text{Suspension in } \mathbb{R}^n & & \\ & y_{t+1} = f(y_t) & \longrightarrow & y' = p(t,y) & \\ \text{Encoding in } \mathbb{N}^n & \nwarrow & & \swarrow & \text{computation of } y \\ & & \text{TM} & & \end{array}$$

- TM: Turing machine
- Discrete dynamical system:
  $y_0 = x$, $y_{t+1} = f(y_t)$, $x \in \mathbb{N}^n$, $f : \mathbb{N}^n \to \mathbb{N}^n$, $t \in \mathbb{N}$
- Continuous dynamical system:
  $y(0) = x$, $y' = p(t, y)$, $x \in \mathbb{R}^n$, $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $t \in \mathbb{R}_0^+$

Definition (Dynamical systems)

A discrete dynamical system $[X, \mathbb{N}, \phi]$ defined on the topological space X over $\mathbb{N}$ is a function $\phi : X \times \mathbb{N} \to X$ with the properties:

1. initial condition: $\phi(x, 0) = x$ for all $x \in X$;
2. continuity on both arguments;
3. semigroup property: $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{N}$

A continuous dynamical system $[X, \mathbb{R}_0^+, \phi]$ is defined analogously, where $\mathbb{N}$ is replaced by $\mathbb{R}_0^+$.

Observations:

- A discrete dynamical system can be written as $y_{t+1} = f(y_t)$, with $y_0 = x$, where $f(.) = \phi(., 1)$ is called the transition function.
- If ϕ is continuously differentiable with respect to t then, a continuous dynamical system gives rise to an initial value problem $y' = f(t, y)$, $y(0) = x$, where f is called a vector field.

Some results:

- On $[0, 1] \times [0, 1]$ there is a piecewise linear function that simulates the transition function of an arbitrary TM (Moore 1991)
- On $\mathbb{R}$ there is an analytic closed form function that simulates the transition function of an arbitrary TM (Moore and Koiran 1996)
- 3-dimensional piecewise constant differential flows on bounded domains simulate the dynamics of an arbitrary TM (Asarin, Maler and Pnueli 1995)
- On $\mathbb{R}^3$ there is a continuous flow $y' = f(t, y)$ that simulates the dynamics of an arbitrary TM (Branicky 1996)
- Several authors restrict f in $y' = f(t, y)$ to certain classes of functions, which can be in particular C^k or C^∞.
- It is conjectured that no analytic map on a compact, finite-dimensional space can simulate a Turing machine through a reasonable encoding (Moore 1998).

Encoding TMs

Encoding Turing machine as a discrete dynamical systems on $\mathbb{N}^3$:

$$\begin{array}{ccc} (\omega, s, p) & \xrightarrow{\psi} & x \in \mathbb{N}^3 \\ \delta \downarrow & & \downarrow f \\ (\omega', s', p') & \xrightarrow{\psi} & f(x) \in \mathbb{N}^3 \end{array}$$

If the tape contents is

$$\ldots 000 a_{-p} \ldots a_{-1} a_0 a_1 \ldots a_n 000 \ldots$$

$\omega \in \{1, \ldots, 9\}^*$, $s \in \{1, \ldots, m\}$, then

$$\psi : \begin{cases} x_1 = a_0 + a_1\, 10 + \cdots + a_n\, 10^n \\ x_2 = a_{-1} + a_{-2}\, 10 + \cdots + a_{-p}\, 10^{p-1} \\ x_3 = s \end{cases}$$

Extension

Definition

The map $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ is a robust extension of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there are $\delta_{in}, \delta_{ev}, \delta_{out} \in (0, \frac{1}{2})$ such that

$$||n_0 - x_0||_\infty \leq \delta_{in}, \ ||\Omega - \overline{\Omega}||_\infty \leq \delta_{ev}$$

implies that

$$||\omega^{[k]}(n_0) - \overline{\Omega}^{[k]}(x_0)||_\infty \leq \delta_{out}$$

for all $k \in \mathbb{N}$

Extensions for TMs

Definition (PIVP function)

A PIVP function with parameters in S is the solution of the polynomial initial value problem $y' = p(t, y),\ y(0) = y_0$ where the coefficients of p and the components of y_0 are in S

Definition ($\mathbb{Q}[\pi]$)

$\mathbb{Q}[\pi] := \{a_n\pi^n + \cdots + a_1\pi + a_0 \in \mathbb{R} \mid a_0, \ldots, a_n \in \mathbb{Q}\}$.

Theorem (Graça, Campagnolo and Buescu, 2008)

The transition function $\omega : \mathbb{N}^3 \to \mathbb{N}^3$ of a Turing machine (under the encoding ψ) admits a robust extension $\Omega : \mathbb{R}^3 \to \mathbb{R}^3$.
Ω can be chosen to be a composition of polynomials with coefficients in $\mathbb{Q}[\pi]$ and PIVP functions with parameters in $\mathbb{Q}[\pi]$.

Suspension

Definition (Suspension)

The solution ϕ of $y' = f(t, y)$, $y(0) = n_0 \in \mathbb{N}^m$

is a robust suspension of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there are $\delta_{in}, \delta_{ev}, \delta_{out}, \delta_{time} \in (0, \frac{1}{2})$ such that

$$||n_0 - y_0||_\infty \leq \delta_{in}, \ ||f - \overline{f}||_\infty \leq \delta_{ev}, \ ||t - k||_\infty \leq \delta_{time}$$

implies that the solution $\overline{\phi}$ of $y' = \overline{f}(t, y)$, $y(0) = y_0$

satisfies

$$||\omega^{[k]}(n_0) - \overline{\phi}(t)||_\infty \leq \delta_{out} \ \text{ for all } k \in \mathbb{N}$$

.

Construction of the suspension

Branicky's clocks:

- The solution of
$$y' = c\,(b - y)^3 \phi(t)$$
approaches at $t = 1$ the "target" b with arbitrary precision, independently of the initial condition at $t = 0$.
- The following system allows to iterate the map Ω

$$\begin{cases} z_1' = c_1(\Omega(r(z_2)) - z_1)^3 \theta(\sin 2\pi t) \\ z_2' = c_2(r(z_1) - z_2)^3 \theta(-\sin 2\pi t) \end{cases} \qquad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0 \end{cases}$$

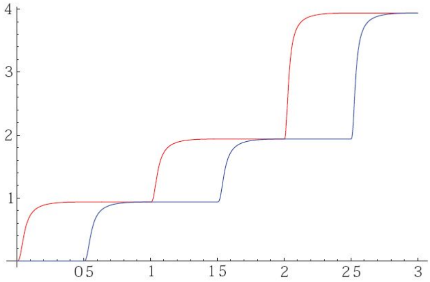

Figure: Simulation of the iteration of $f(n) = 2^n$ with ODEs

Removing the non-PIVP functions

Replacing the rounding function r and the control functions $\theta(\sin 2\pi t)$ by appropriate PIVP functions, and applying a PIVP error-contracting function a number of times depending on n, we obtain an approximation result on the integers:

$$|z(x_1, x_2, t) - z^*(x_1, x_2, t, n)| < \frac{1}{n}.$$

This is enough for our needs since we just want a suspension of a dynamical system on $\mathbb{N}^m$.

Suspension for discrete dynamical suystems with PIVP functions

Theorem (Graça, Buescu and Campagnolo, to appear)

If the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ admits a robust extension $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ whose components are compositions of polynomials and PIVP functions with parameters in $\mathbb{Q}[\pi]$, then ω admits a robust suspension ϕ which a PIVP function with parameters in $\mathbb{Q}[\pi]$.

Corollary

The transition function $\omega : \mathbb{N}^3 \to \mathbb{N}^3$ of a Turing machine (under the encoding ψ) admits a robust suspension ϕ. Moreover ϕ is a PIVP function with parameters in $\mathbb{Q}[\pi]$.

Reachability

Corollary

The following problem is undecidable:

> *Given a vector of polynomial* $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *with coefficients in* $\mathbb{Q}[\pi]$, $y_0 \in \mathbb{Q} \times \mathbb{Q}^n$, *and an open set* A *in* $\mathbb{R}^n$ *decide if the solution of*
>
> $$y' = p(t, y), \; y(0) = y_0$$
>
> *crosses* A.

Boundedness of the maximal interval of existence

Note: the maximal interval of the PIVP

$$y' = \alpha(y^2 - 1)t, \; y(0) = 3$$

is bounded for $\alpha > 0$ and unbounded for $\alpha \leq 0$.

Unless one can decide $\alpha \neq 0$ this gives rise to trivial undecidability results.

Let's restrict the parameters of the PIVP to $\mathbb{Q}[\pi]$, which is a comparable set, i.e., given $\alpha, \beta \in \mathbb{Q}[\pi]$ we can decide if $\alpha = \beta$ and $\alpha < \beta$.

Theorem (Graça, Buescu and Campagnolo, to appear)

The following problem is undecidable:

> *Given a vector of polynomial* $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *with coefficients in* $\mathbb{Q}[\pi]$ *and* $y_0 \in \mathbb{Q}^n$*, decide if the maximal interval of existence of the solution of* $y' = p(t, y),\ y(0) = y_0$ *is bounded.*

Proof.

Let x_q be the component of the PIVP that encodes the state in a TM suspension. Consider the system (equivalent to a PIVP)

$$z_1' = x_q - (m - \frac{1}{2}), \quad z_2 = \frac{1}{z_1}, \quad z_1(0) = z_2(0) = -1.$$

The TM halts when x_q reaches m; then z_1' becomes larger than 1/8 and, eventually, z_2 blows up. Otherwise, z_1 always decreases and the solution is defined everywhere. □

Approximating real computable functions

Theorem

Let $f : [a,b] \to \mathbb{R}$ be a computable function. Then there exists a PIVP such that if we set the initial conditions $(x, \bar{n}) \in [a,b] \times \mathbb{R}$, where $|\bar{n} - n| \leq \varepsilon < 1/2$, with $n \in \mathbb{N}$, there exists some $T \geq 0$ such that the solution of the PIVP satisfies $|y_i(t) - f(x)| \leq 2^{-n}$ for all $t \geq T$.

We can use this to simulate an oracle Turing machine that computes the function f.

Approximations

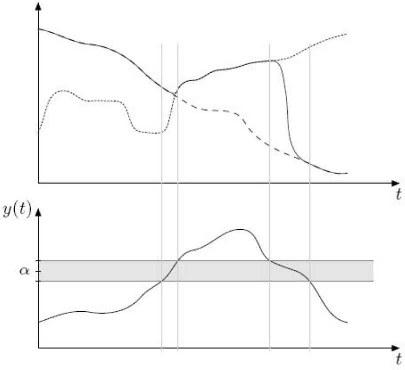

Figure: Switching functions f_1 and f_2 with a control function y

Theorem (see Bournez, Campagnolo, Graça and Hainry, 2007)

On computable compact intervals:

- *(Approximation)* $\mathbf{C}(\mathbb{R}) \approx GPAC_{\mathbb{Q}[\pi]}$
- *(Completion)* $\mathbf{C}(\mathbb{R}) = GPAC_{\mathbb{Q}[\pi]}(LIM^*)$

Summary

- There are PIVPs able to simulate arbitrary Turing machines (on unbounded domains);
- Although the solutions of PIVPs with computable coefficients are computable, several properties of those dynamical systems are undecidable, even if the parameters of the system are in $\mathbb{Q}[\pi]$.
- PIVPs, which are a well known model of physical phenomena, are also a robust yet powerful model of continuous time computation.
- Can we get rid of π and obtain the result about suspensions for PIVPs with parameters in $\mathbb{Q}$?
- Can we do a more genuine suspension of discrete dynamical systems, which doesn't rely on "clocks"?

Some references

- D.S. Graça, J. Buescu, and M.L. Campagnolo. Computational bounds on polynomial differential equations, Applied Mathematics and Computation, to appear.
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- E. Asarin and A. Bouajjani. Perturbed Turing machines and hybrid systems. In Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science (LICS-01), Los Alamitos, CA. IEEE Computer Society Press (2001) 269:278.
- C. Moore, Finite-dimensional analog computers: flows, maps and recurrent networks, in C. Calude, J. Casti and M. Dinneen (Eds) Proceedings of UMC'98, Springer, (1998), 59:71

Real recursive functions

Proposed by C. Moore; it can be seen as a further generalizations of the GPAC.

It allows to:

- Simplify classes using approximations;
- Lift functions from $\mathbb{N}$ to $\mathbb{R}$;
- Derive machine independent characterizations of analog classes (some will correspond to the real computable functions).