A tutorial on Analog Computation

Computing functions over the reals
III

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Tutorial outline:

- I Computing real functions
- II Continuous dynamical systems and computation
- III Real recursive functions

Outline

Function Algebras and Real Recursive Functions

Definition

A Function Algebra $\mathrm{FA}[f_1, \ldots, f_k; \mathrm{op}_1, \ldots, \mathrm{op}_n]$ is the smallest set of functions containing $f_1, \ldots, f_k$, and closed under the operations $\mathrm{op}_1, \ldots, \mathrm{op}_n$.

Example: $\mathrm{FA}[0, 1, +, \dot{-}, \mathrm{P}; \mathrm{comp}, \mathrm{BSUM}, \mathrm{BPROD}, \mu]$, which is equal to the computable functions over the naturals.

Real Recursive Functions: Function algebras over the reals, introduced by C. Moore 1996.

Some basic functions over the reals

- Constant functions: $0, 1, -1$
- Projection functions "P" (example: $\mathrm{P}(x, y) = x$)
- $\theta_k(x) = \begin{cases} 0, & x < 0; \\ x^k, & x \geq 0. \end{cases}$

The Differential Equation Operation

Definition (The operation ODE)

- **Input**: $g(x)$, $f(y, u, x)$.
- **Output**: The solution of the IVP

$$h(0, x) = g(x), \qquad \frac{\partial}{\partial y} h = f(y, h, x)$$

Definition (The operation LI)

LI is the operation defined like ODE, except that f must be linear in h.

Two real function algebras and some examples

Definition ($\mathcal{L}_k$)

$\mathcal{L}_k = \mathrm{FA}[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{LI}]$

Definition ($\mathcal{ODE}_k^\star$)

It is the class of total functions of $\mathrm{FA}[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{ODE}]$

Some functions in $\mathcal{L}_k$: e^x, $(x + y)$, (xy), $\sin x$, $\ldots$

Some functions in $\mathcal{ODE}_k^\star$: $\ln(x^2 + 1)$, $\arctan(x)$, $\ldots$

A few constructions and the corresponding differential equations:

- $f(x,y) = x + y$ $\quad$ $f(x,0) = x$, $\partial_y f(x,y) = 1$ $\quad$ $f' = [1\ 1]$

- $f(x,y) = xy$ $\quad$ $f(x,0) = 0$, $\partial_y f(x,y) = x$ $\quad$ $f' = [y\ x]$

- $(\sin y, \cos y)$ $\quad$ $f(0) = (0,1)$, $\partial_y f(y) = (\cos y, -\sin y)$ $\quad$ $\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix}$

Some nice features of functions algebras:

- it is easy to “tune” the classes, adding or removing basic functions and operations;
- their properties can be proved by induction;

Real recursive function theory can be seen as a generalization of systems of differential equations:

- PIVP functions correspond to unary functions that can be defined from real constants, $+$, $\times$ and projections, and closure under comp and ODE (Graça and Costa 2003).

A characterization of $\mathbf{E}(\mathbb{N})$

$\mathbf{E}(\mathbb{N})$ denotes Kalmar's elementary functions, which are closed under composition, bounded sums and bounded products.

Definition (Discrete part)

$\mathrm{dp}(\mathcal{F}) = \{f|_{\mathbb{N}} \text{ s.t. } f \in \mathcal{F} \text{ and } f(\mathbb{N}) \subset \mathbb{N}\}$

Theorem (Campagnolo, Moore and Costa 2002)

$dp(\mathcal{L}_k) = \mathbf{E}(\mathbb{N})$

Incidentally, this means that "cascades" of linear differential equations have a large computational power.

A characterization of $\mathbf{E}(\mathbb{R})$

Bournez and Hainry propose a characterization of the elementary computable real functions.

Definition (The operation LIM_ω)

- **Input**: $f(\bar{x}, t)$ s.t. its derivative $\|\frac{\partial}{\partial t} f\|$ decreases exponentially with t
- **Output**: $F(\bar{x}) = \lim_t \to \infty f(\bar{x}, t)$ if F is of class C^2.

Theorem (Bournez and Hainry 2005)

For C^2 functions on compact domains,
$\mathbf{E}(\mathbb{R}) = FA[0, 1, -1, \theta_3, P; comp, LI, LIM_\omega]$

They also propose a characterization of $\mathbf{C}(\mathbb{R})$, for C^2 functions and compact domains, using an additional restricted minimalization operator (Bournez and Hainry, 2006).

Approximation

Recall that ...

Definition

- $f(\bar{x}) \preceq f^*(\bar{x}, t)$ means $|f(\bar{x}) - f^*(\bar{x}, t)| < \dfrac{1}{t}$ for all $t > 0$.
- For classes of functions $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}$ means:

  For any $f \in \mathcal{A}$ there is $f^* \in \mathcal{B}$ such that $f \preceq f^*$.

Lemma (Transitivity)

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are classes of functions and suppose $\mathcal{C}$ contains $+$, id and is closed under composition. Then $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{C}$ implies $\mathcal{A} \preceq \mathcal{C}$.

The Limit Operation

and that ...

Definition (The operation LIM)

- **Input**: $f^*(t, \bar{x})$
- **Output**: $f(\bar{x}) = \lim_{t \to \infty} f(t, \bar{x})$, if the limit exists, and $f \preceq f^*$.

Definition

If $\mathcal{F}$ a set of functions, then $\mathcal{F}(\text{LIM})$ is $\mathcal{F}$ closed under the operation LIM.

$\mathbf{E}(\mathbb{R})$ and $\mathcal{L}$

$\mathcal{L}_k$ denotes FA$[0, 1, -1, \pi, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{LI}]$.
Let $\mathcal{L}^{\mathrm{a}}$ abbreviate FA$[0, 1, -1, \mathrm{P}; \mathrm{comp}, \mathrm{LI}]$.

Theorem (Campagnolo and Ojakian, 2008)

- *(Approximation)* $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}_k \approx \mathcal{L}^{\mathrm{a}}$
- *(Completion)* $\mathbf{E}(\mathbb{R}) = \mathcal{L}_k(LIM) = \mathcal{L}^{\mathrm{a}}(LIM)$
- *(Alternative Completion)* $\mathbf{E}(\mathbb{R}) = \mathcal{L}_k(dLIM)$ *(similar to Bournez and Hainry 2005)*

Definition (The operation dLIM)

- Input: $f(t, \bar{x})$
- Output: $F(\bar{x}) = \lim_{t \to \infty} f(t, \bar{x})$, if $|\frac{\partial}{\partial t} f| \leq 1/2^t$ for $t \geq 1$.

Sketch of the proof of $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}$

Idea: lift the equality from $\mathbb{N}$ to $\mathbb{R}$

$$
\begin{array}{ccc}
\text{TM models} & & \text{Function algebras} \\
\mathbf{E}(\mathbb{N}) & = & \mathrm{FA}_{\mathbb{N}} \\
\preceq \quad \succeq & & \preceq \quad \succeq \\
\widetilde{\mathbf{E}(\mathbb{Q})} & = & \widetilde{\mathrm{dFA}_{\mathbb{Q}}} \\
\wr\wr & & \wr\wr \\
\mathbf{E}(\mathbb{R}) & & \mathrm{FA}_{\mathbb{Q}} \\
 & & \wr\wr \\
 & & \mathcal{L}
\end{array}
$$

Figure: Approximations to prove $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}$

The simplification $\mathcal{L} \approx \mathcal{L}^{\mathrm{a}}$

Recall:

$$\mathcal{L}_k = FA[0, 1, -1, \pi, \theta_k, P; comp, LI]$$
$$\mathcal{L}^{\mathrm{a}} = FA[0, 1, -1, P; comp, LI].$$

Goal: eliminate the non-analytic function θ_k.

Show:

- $\theta_k, \pi \preceq \mathcal{L}^{\mathrm{a}}$
- comp, LI $\preceq \mathcal{L}^{\mathrm{a}}$

General idea: using approximation and transitivity we can break down the proof of $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}^{\mathrm{a}}$ into simpler pieces.

The computable real functions $C(\mathbb{R})$

Definition (The operation CLI)

CLI is similar to LI, except that the output has to be bounded by a function in the class

Definition (The operation UMU)

- **Input**: $f(t, \bar{x})$ such that
  1. For any $\bar{x}$, $f(t, \bar{x})$ increases in t, and
  2. For any $\bar{x}$, there is a unique T such that $f(T, \bar{x}) = 0$ (and at that T, $\frac{\partial}{\partial t} f > 0$).
- **Output**: Function $F(\bar{x})$ = the unique T such that $f(T, \bar{x}) = 0$.

We can write in terms of approximation and completion a characterization of the real computable functions:

Theorem (similar to Bournez and Hainry, 2006)

For C^2 functions on compact intervals:

- *(Approximation)* $\mathbf{C}(\mathbb{R}) \approx FA[0, 1, \theta, P; comp, CLI, UMU]$
- *(Completion)*

$$\begin{aligned} \mathbf{C}(\mathbb{R}) &= FA[0, 1, \theta, P; comp, CLI, UMU](LIM) \\ &= FA[0, 1, \theta, P; comp, CLI, UMU](dLIM) \end{aligned}$$

Another improvement

Definition (a FA based on zero-finding)

Let $\mathcal{UMU}_k$ be FA$[0, 1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{LI}, \mathrm{UMU}]$

Recall ...

Definition (a FA based on ordinary differential equations)

Let $\mathcal{ODE}_k^\star$ be FA$[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{ODE}]$

An improved characterization of the real computable functions:

Theorem

For $k \geq 2$, $\mathbf{C}_\mathbb{R} = \mathcal{ODE}_k^\star(LIM) = \mathcal{UMU}_k(LIM)$.

Note: the compactness and C^2 requirements are dropped.

Discussion of the proof

Main step of the proof: $\mathbf{C}_{\mathbb{R}} \subseteq \mathcal{UMU}_k(\mathrm{LIM})$

Goal: $\mathbf{C}_{\mathbb{R}} \preceq \mathcal{UMU}_k$ Implies: $\mathbf{C}_{\mathbb{R}} \subseteq \mathcal{UMU}_k(\mathrm{LIM})$

The approach: avoid Turing Machine simulations

- overcome unnecessary restrictions;
- appears more general.

Breaking up the proof

$$
\begin{array}{ccc}
\text{Turing machine models} & & \text{Function algebras} \\
\widetilde{\mathbf{dC}_{\mathbb{Q}}} & \subseteq & \widetilde{\mathbf{d}\mathcal{MU}_{\mathbb{Q}}} \\
\rotatebox{90}{$\preceq$} & & \rotatebox{90}{$\succeq$} \\
\mathbf{C}_{\mathbb{Q}} & & \mathcal{MU}_{\mathbb{Q}} \\
 & & \rotatebox{90}{$\succeq$} \\
 & & \mathcal{UMU}_k
\end{array}
$$

Figure: Approximations used to prove $\mathbf{C}_{\mathbb{Q}} \preceq \mathcal{UMU}_k$.

A function algebra for the computable functions

Definition

Let $\mathcal{MU}_{\mathbb{N}}$ be FA$[0, 1, +, \dot{-}, \mathrm{P}; \mathrm{comp}, \mathrm{BSUM}, \mathrm{BPROD}, \mathrm{MU}]$.

Definition

The operation MU. **Input:** $f(t, \bar{x})$ (over $\mathbb{N}$) satisfying:

For each $\bar{x}$, there is a unique $T \geq 1$ such that $f(T, \bar{x}) = 0$, and otherwise

$$f(t, \bar{x}) = \begin{cases} -1, & \textit{if } t < T; \\ 1, & \textit{if } t > T. \end{cases}$$

Output: The function $g(\bar{x}) =$ the unique T such that $f(T, \bar{x}) = 0$.

A function algebra over $\mathbb{Q}$

Definition

Let $\mathbf{C}_{\mathbb{Q}}$ be $\{f_{|\mathbb{Q}} \mid f \in \mathbf{C}_{\mathbb{R}}\}$

Definition

Let $\mathcal{MU}_{\mathbb{Q}}$ be
$\mathrm{FA}[0, 1, -1, +, \mathrm{P}, *, \mathrm{div}, \theta_1; \mathrm{comp}, \mathrm{BSUM}_{\mathbb{Q}}, \mathrm{BPROD}_{\mathbb{Q}}, \mathrm{MU}_{\mathbb{Q}}, \mathrm{Lin}_{\mathbb{Q}}]$.

Definition

Suppose OP takes a function $f : \mathbb{N}^k \to \mathbb{N}$ and returns a function $g : \mathbb{N}^m \to \mathbb{N}$. Then $\mathrm{OP}_{\mathbb{Q}}$ is the following operation:

1. $\mathrm{OP}_{\mathbb{Q}}$ takes as input $f : \mathbb{Q}^k \to \mathbb{Q}$ such that $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$.
2. $\mathrm{OP}_{\mathbb{Q}}$ then applies OP to $f_{|\mathbb{N}}$ to get some $g : \mathbb{N}^m \to \mathbb{N}$.
3. $\mathrm{OP}_{\mathbb{Q}}$ outputs $\mathrm{Lin}_{\mathbb{Q}}(g)$.

The inductive proof

Proof proceeds inductively on the function algebra $\mathcal{MU}_{\mathbb{Q}}$:

- Show the basic functions of $\mathcal{MU}_{\mathbb{Q}}$ are approximated by $\mathcal{UMU}_k$.
- Show that the operations of $\mathcal{MU}_{\mathbb{Q}}$ preserve the approximation:

  For $f \in \mathcal{MU}_{\mathbb{Q}}$ and $f^* \in \mathcal{UMU}_k$, suppose $g = OP(f)$, and $f \preceq f^*$. Then there is $g^* \in \mathcal{UMU}_k$ such that $g \preceq g^*$.

Summary

- Computable Analysis can be caracterized with real recursive functions.
- The connections can be organized using approximation and completion.
- New useful techniques: transitivity, eliminating non-analytic functions, lifting.

Some references

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Overview

- Goal: classify models that compute over $\mathbb{R}$
  - to understand their computational power;
  - to understand their limitations.
- If we allow "completion", approximation is the key to compare different models of computation over $\mathbb{R}$;
- Could alternative models facilitate technical work? (e.g. like showing a function or operation is or is not computable)
- Some equivalences between models emerge;

A general goal

Quest for a Church-Turing type thesis for computation on the reals:

- There are many distinct models of computation on the reals: Computable Analysis, Real recursive Functions, General Purpose Analog Computer, Neural Networks, Dynamic Systems,...
- How are the models distinct?
- What kind of modifications make them equal?
- Are there characterizations of Computable Analysis, which naturally capture all of its functions, *without* a completion operation?